

## Stability in Generalized Modified Gravity

Guido Cognola<sup>a</sup>, Lorenzo Sebastiani<sup>a</sup>, Sergio Zerbini<sup>a</sup>

<sup>a</sup> *Dipartimento di Fisica, Università degli Studi di Trento,  
Trento, 38100, Italy*

*INFN - Gruppo Collegato di Trento*

The stability issue of Generalized modified gravitational models is discussed with particular emphasis to de Sitter solutions. Two approaches are briefly presented.

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### 1. The de Sitter stability issue

It is well known that recent astrophysical data are in agreement with a universe in current phase of accelerated expansion, in contrast with the predictions of Einstein gravity in FRW space-time. Most part of energy contents, roughly 75 % in the universe is due to mysterious entity with negative pressure: Dark Energy. The simplest explanation is Einstein gravity plus a small positive cosmological constant. As an alternative, one may consider more drastic modification of General Relativity: Modified Gravity Models, see for example.<sup>1-4</sup>

We shall deal with modified generalized models, described by a Lagrangian density  $F(R, P, Q)$ ,<sup>5</sup> where  $R$  is the Ricci scalar, and  $P = R_{\mu\nu}R^{\mu\nu}$ , and  $Q = R_{\mu\nu\alpha\beta}R^{\mu\nu\alpha\beta}$  are quadratic curvature invariants. In particular the Gauss-Bonnet topological invariant reads  $G = R^2 - 4P + Q$ . The stability of the de Sitter solution, relevant for Dark energy, may be investigated in these Gauss-Bonnet models in several ways. We limit ourselves to the following two approaches: Perturbation of Eq. of Motion in the Jordan frame and Dynamical System Approach in FRW space-time.

### 2. Stability of $F(R, P, Q)$ model in the Jordan frame

The starting point is the trace of the equations of motion, which is trivial in Einstein gravity  $R = -\kappa^2 T$ , but, for a general  $F(R, P, Q)$  model, reads

$$\nabla^2 (3F'_R + RF'_P) + 2\nabla_\mu \nabla_\nu [(F'_P + 2F'_Q) R^{\mu\nu}] - 2F + RF'_R + 2(F'_P + F'_Q) = \kappa^2 T. \quad (1)$$

Requiring  $R = R_0 = Cts$ ,  $P_0 = Cts$ , and  $Q_0 = Cts$  one has de Sitter existence condition in vacuum

$$2F_0 - R_0 F'_{R_0} - 2P_0 F'_{P_0} - 2Q'_0 = 0. \quad (2)$$

Perturbing around dS space, namely  $R = R_0 + \delta R$ , and with  $P = P_0 + \delta P$ , and  $Q = Q_0 + \delta Q$ , observing that  $\delta P = \frac{R_0}{2}\delta R$ , and  $\delta Q = \frac{R_0}{6}\delta R$ , one arrives at the perturbation Eq.

$$-\nabla^2 \delta R + M^2 \delta R = 0, \quad (3)$$

in which the scalaron effective mass reads

$$M^2 = \frac{R_0}{3} \left( \frac{F'_{R_0} + 4H_0^2 \left( \frac{3}{2} F'_{P_0} + F'_{Q_0} \right)}{R_0 [A_R + A_Q + A_P + 4H_0^2 \left( \frac{3}{2} F'_{P_0} + F'_{Q_0} \right)]} - 1 \right) \quad (4)$$

where

$$A_R = F''_{R_0 R_0} + 6H_0^2 F''_{R_0 P_0} + 4H_0^2 F''_{R_0 Q_0} \quad (5)$$

$$A_Q = 2H_0^2 (F''_{R_0 P_0} + 6H_0^2 F''_{P_0 P_0} + 4H_0^2 F''_{P_0 Q_0}) \quad (6)$$

$$A_P = 4H_0^2 (F''_{R_0 Q_0} + 6H_0^2 F''_{Q_0 P_0} + 4H_0^2 F''_{Q_0 Q_0}) \quad (7)$$

Thus, if  $M^2 > 0$ , one has stability of the dS solution. In the particular case  $F(R, G)$ , one has<sup>6,7</sup>

$$\frac{9F'_{R_0}}{R_0 [9F''_{R_0 R_0} + 6R_0 F''_{R_0 G_0} + R_0^2 F''_{G_0 G_0}]} > 1. \quad (8)$$

In the case of a  $F(R)$  models, one has the well known condition  $\frac{F'_{R_0}}{R_0 F''_{R_0}} > 1$ .

### 3. Dynamical System Approach

This approach has been used by many authors. One works in a cosmological setting, namely with a FRW metric, and the main idea consists in rewriting the generalized Einstein-Friedman equations in an equivalent system of first order differential equations, introducing new dynamical variables  $\Omega_i$

$$\frac{d}{dt} \vec{\Omega}(t) = \vec{v}(\vec{\Omega}(t)). \quad (9)$$

Here the evolution parameter has been denoted by  $t$ . The critical (or fixed) points are defined by  $\vec{v}(\vec{\Omega}_0) = 0$ . The key point is:

**Hartman-Grobman theorem:** *The orbit structure of a dynamical system in the neighbourhood of a hyperbolic fixed point is topologically equivalent to the orbit structure of the associated linearized dynamical system, defined by a stability matrix  $M_0$ .*

Thus, in order to study the stability of the above non linear system of differential Eqs. at critical points, it is sufficient to investigate the related linear system of differential Eqs.:

$$\frac{d}{dt} \delta \vec{\Omega}(t) = M_0 \delta \vec{\Omega}(t), \quad M_0 \text{ Jacobian matrix evaluated at } \vec{\Omega}_0 \quad (10)$$

The solution of the linearization is well known and the evolution is determined by the signs of the eigenvalues of  $M_0$ . As a result, the non linear system is stable if all eigenvalues of the matrix  $M_0$  have negative real parts.

As an example, let us consider a modified model  $R + f(G)$ . The related autonomous system in the two unknown quantities  $G$  and  $H$  reads

$$\dot{G} = \frac{1}{24f''_G H^3} ((Gf'_G - f) - 6H^2), \quad \dot{H} = \frac{G}{24H^2} - H^2. \quad (11)$$

The critical points are defined by  $\dot{G} = 0$  and  $\dot{H} = 0$ . Thus, we have the solutions  $24H_0^4 = G_0$  and  $G_0 f'_0 - f_0 = 6H_0^2$  and these correspond to a de Sitter critical point with Gauss-Bonnet invariant. The linearized system around de Sitter critical point reads

$$\delta \dot{G} = H_0 \delta G - \frac{1}{2H_0^2 f''_0} \delta H, \quad \delta \dot{H} = \frac{\delta G}{24H_0^2} - 4H_0 \delta H. \quad (12)$$

One can read off the stability matrix and the stability condition is  $\frac{9}{R_0^3 f''_0} > 1$ , in agreement with the previous approach.

#### 4. Concluding remarks

Modified gravity may be seen as the phenomenological description of a fundamental unknown theory. From this point of view, corrections to Einstein-Hilbert action depending on higher order curvature invariants are likely to be expected (Lovelock gravity is an example).

Among many existing approaches, two methods have been illustrated in order to investigate the stability of these models around de Sitter critical points, and the dS stability conditions has been derived within two possible approaches.

These methods have owns advantages and problems, and, in our opinion, both permit to study critical points and stability for modified gravitational models depending on arbitrary geometric invariants, generalising the results obtained for  $F(R)$  models with other methods (see, for example<sup>8-10</sup>). We conclude noting that the dynamical system approach has also been applied to non local  $F(R)$  models.<sup>11,12</sup>

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